NOTATION

I, electric current; l, f, π, respectively, the length, the cross-sectional area, and the perimeter of a metal rod; z, longitudinal coordinate; λ, thermal conductivity; ρ, electrical resistivity; a, b, coefficients in the temperature dependence of electrical resistivity; q_s, specific thermal flux; ϑ, temperature drop; and k, wave number.

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APPROXIMATE SOLUTION OF THE PROBLEM OF SOLIDIFICATION OF CURVILINEAR WALLS AND HOLLOW BODIES UNDER TRANSIENT CONDITIONS

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An approximate solution is obtained for the problem of solidification of curvilinear walls, hollow and continuous bodies in a transient process under conditions of operation of a set of thermal regime factors.

The operation of a large number of different instrumentation objects is characterized by a pulse heat liberation law in combination with high densities of the thermal fluxes being dissipated. This circumstance governed the broad application of melting materials and heat accumulators, in problems to support the thermal regimes of apparatus [1]. Unfortunately, the solution of nonstationary heat conduction problems associated with the phase transition process evokes substantial difficulties of a calculational nature. For this reason, the majority of known analytic solutions of the solidification problems includes bodies of canonical form (plates, cylinders, spheres) [2]. Of considerably greater interest in engineering is the possibility of analyzing the dynamics of hollow bodies and curvilinear shells, as well as of bodies of complex configuration.

Let us consider the problem of shell solidification (Fig. 1) characterized by the governing dimensions R_1 and R_2. Integral geometric properties of the shell are given by values of the inner S_1 and outer S_2 surfaces and the corresponding volumes V_1 and V_2.

The solidification process is determined by the action of two groups of regime factors:

1) actions from the surface S_1 of a medium with temperature t_1 and a heat source of density q_1 distributed uniformly over this surface;

2) actions from the surface S_2 of a medium with temperature t_2 and a surface heat source of density q_2.

The intensity of the heat exchange process on surfaces S_1 and S_2 is given by the heat transfer coefficients α_1 and α_2. The thermophysical characteristics (heat conduction coefficients λ_1 and λ_2, densities γ_1 and γ_2, heat of phase transition ρ) of the solid 1 and liquid 2 phases are considered given.

At the initial instant $\tau = 0$ the temperature of the medium t_2 is reduced to a temperature less than the solidification temperature t_3, and later remains constant.

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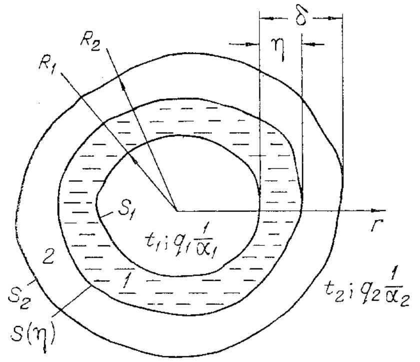

Fig. 1. Shell under solidification conditions in the solid (1) and liquid (2) phases.

The heat exchange condition on the phase interface has the form [2]

$$\lambda_1 \frac{\partial t_1(r,\ \tau)}{\partial r} - \lambda_2 \frac{\partial t_2(r,\ \tau)}{\partial r} = \rho \gamma_1 \frac{d\eta}{d\tau}. \tag{1}$$

We use the first method of L. S. Leibenzon, based on the selection of temperature functions satisfying the heat exchange boundary conditions, to solve the problem formulated.

The possibility of analyzing the temperature field in bodies of complex shape on the basis of a generalized one-dimensional heat conduction equation was displayed by N. A. Yaryshev [3]. This aspect was later developed in [4-6].

In conformity with [6], the stationary temperature field of a curvilinear wall has the form

$$t(\bar{r}) = Z_1 B_0(\bar{r}) + Z_2 C_0(\bar{r}), \tag{2}$$

where Z_1 and Z_2 are the generalized thermal effects

$$Z_1 = t_1 + q_1 \frac{1}{\alpha_1}, \quad Z_2 = t_2 + q_2 \frac{1}{\alpha_2}. \tag{3}$$

The coefficients $B_0(\bar{r})$ and $C_0(\bar{r})$ have the form

$$B_0(\bar{r}) = \frac{1}{E}\left[\varphi_1\left(1 + \frac{2}{\xi_2}\right) + \varphi_2 - 2\varphi_2 \bar{r} + (\varphi_2 - \varphi_1)\bar{r}^2\right],$$

$$C_0(\bar{r}) = \frac{1}{E}\left[2\varphi_2 \frac{1}{\xi_1} + 2\varphi_2 \bar{r} + (\varphi_1 - \varphi_2)\bar{r}^2\right], \tag{4}$$

$$E = \varphi_1\left(1 + \frac{2}{\xi_2}\right) + \varphi_2\left(1 + \frac{2}{\xi_1}\right), \quad \bar{r} = \frac{r}{\delta}, \quad \delta = R_2 - R_1, \tag{5}$$

where φ_1, φ_2 are complexes computed from the formulas

$$\varphi_1 = \delta \frac{S_1}{V_1}, \quad \varphi_2 = \delta \frac{S_2}{V_2}; \tag{6}$$

and ξ_1, ξ_2 are the Biot numbers governing the heat-exchange intensity

$$\xi_1 = \alpha_1 \frac{\delta}{\lambda_1}, \quad \xi_2 = \alpha_2 \frac{\delta}{\lambda_2}. \tag{7}$$

Describing the temperatures of the liquid $t_1(\bar{r})$ and solid $t_2(\bar{r})$ phases by using (2) and performing intermediate manipulations associated with substituting the expressions obtained into (1) and then differentiating, we obtain

$$\frac{2\lambda_1 S_1\left(t_1 - t_3 + q_1 \frac{1}{\alpha_1}\right)}{S_1 \eta + S(\eta)\left(\eta + \frac{2\lambda_1}{\alpha_1}\right)} + \frac{2\lambda_2 S_2\left(t_2 - t_3 + q_2 \frac{1}{\alpha_2}\right)}{S_2(\delta - \eta) + S(\eta)\left(\delta - \eta + \frac{2\lambda_2}{\alpha_2}\right)} = \rho \gamma_1 \frac{d\eta}{d\tau}, \tag{8}$$

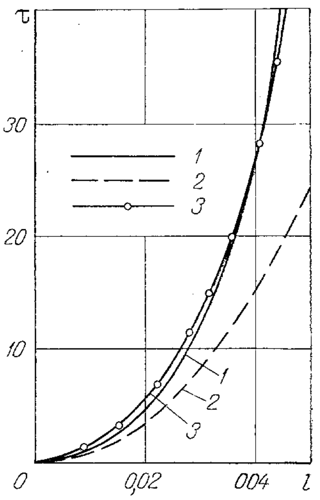

Fig. 2. Solidification time of a layer of palmitic acid: 1) exact solution (14); 2) approximate solution [2], (22); 3) proposed solution (19).

which should be integrated under the initial condition

$$\eta(\tau)|_{\tau=0} = \delta. \tag{9}$$

The solution of (8) can be obtained if the form of the functional dependence $S(\eta)$ is known.

The question of the analytical description of the interrelation between the temperature averaging surface of a one-dimensional solid and the generalized coordinate was examined in [4]. Following the recommendations of this paper, we use the representation

$$S(\eta) = A(R_1 + \eta)^n. \tag{10}$$

The numerical value of the magnitude of the form factor n in (10) is computed by means of the formula [4]

$$n = \frac{\lg \frac{S_2}{S_1}}{\lg \left(\frac{V_2}{V_1} \Big/ \frac{S_2}{S_1} \right)}. \tag{11}$$

For shells of canonical shape — plates, cylinders, and spheres — the form factor takes on the respective values 0, 1, 2.

Taking account of (10), equation (8) becomes

$$\Phi(\eta) = \Phi_1(\eta) + \Phi_2(\eta) = \rho\gamma_1 \frac{d\eta}{d\tau}, \tag{12}$$

where

$$\Phi_1(\eta) = \frac{2\lambda_1 R_1^n \left(t_1 - t_3 + q_1 \frac{1}{\alpha_1} \right)}{R_1^n \eta + (R_1 + \eta)^n \left(\delta + \frac{2\lambda_1}{\alpha_1} \right)}, \quad \Phi_2(\eta) = \frac{2\lambda_2 R_2^n \left(t_2 - t_3 + q_2 \frac{1}{\alpha_2} \right)}{R_2^n (\delta - \eta) + (R_1 + \eta)^n \left(\delta - \eta + \frac{2\lambda_2}{\alpha_2} \right)}. \tag{13}$$

Then the desired solution to estimate the solidification time of a layer of thickness $l = \delta - \eta$ is

$$\tau = \rho\gamma_1 \int\limits_{\delta}^{\eta} \frac{1}{\Phi(\eta)} d\eta, \quad \eta \in (\delta;\ \eta_K), \tag{14}$$

where η_K is the critical value of the solidification zone boundary corresponding to the stationary state of a two-component liquid phase–solid phase system.

The numerical value of the quantity η_K is the positive root of the characteristic equation

$$\Phi_1(\eta) + \Phi_2(\eta) = 0. \tag{15}$$

Equation (12) allows direct integration. However, the structure of (13) evokes substantial difficulties of a computational nature for integration in (14).

Taking account of the boundary conditions

$$\left.\frac{1}{\Phi(\eta)}\right|_{\eta=\delta} = \frac{1}{\Phi(\delta)}, \quad \left.\frac{1}{\Phi(\eta)}\right|_{\eta=\eta_K} = \infty, \tag{16}$$

we simplify this problem by approximating the integrand $1/\Phi(\eta)$ in (14) by the approximate dependence

$$\frac{1}{\Phi(\eta)} = \frac{1}{\Phi(\delta)} - \varepsilon \ln\left(\frac{\eta - \eta_K}{\delta - \eta_K}\right), \tag{17}$$

where

$$\varepsilon = \frac{1}{\Phi[\delta - 0.632(\delta - \eta_K)]} - \frac{1}{\Phi(\delta)}. \tag{18}$$

Then the solidification time of a layer l of thickness $\delta - \eta$ for the shell (Fig. 1) is

$$\tau = \rho\gamma_1\left\{\frac{\eta - \delta}{\Phi(\delta)} - \varepsilon(\delta - \eta_K) - \varepsilon(\eta - \eta_K)\left[\ln\left(\frac{\eta - \eta_K}{\delta - \eta_K}\right) - 1\right]\right\}. \tag{19}$$

Results of computing the solidification time of a palmitic acid plate (δ = 0.1 m) under boundary conditions of the first kind ($\alpha_1 = \alpha_2 = \infty$) and the temperature drops $t_1 - t_3 = t_3 - t_2 = 15°K$ are presented in Fig. 2. The data obtained are compared with the results of a computation using (14) and with numerical values obtained by an approximate solution [2].

Solutions of the problem of "freezing of thawed ground" for bodies of canonical shape under the assumption that the liquid-phase temperature equals the solidification temperature are represented extensively in the literature by a computation of the temperature fields and of the duration of the phase transition process. In this case the direct integration of (8) permits the solution to be obtained with the heat-exchange intensity on the outer $\Phi_1(\eta) = 0$ and inner $\Phi_2(\eta) = 0$ shell boundaries taken into account:

For $t_1 = t_3$, $\Phi_1(\eta) = 0$

$$\tau = \frac{\rho\gamma_1}{2\lambda_2 R_2^n\left(t_2 - t_3 + q_2\frac{1}{\alpha_2}\right)}\left[R_2^n\eta\left(\delta - \frac{\eta}{2}\right) + \frac{(R_1+\eta)^{n+1}}{n+1} \times \right.$$

$$\left. \times\left(\frac{2\lambda_2}{\alpha_2} + R_2\right) - \frac{(R_1+\eta)^{n+2}}{n+2} - R_2^n\frac{\delta^2}{2} - \frac{R_2^{n+1}}{n+1}\left(\frac{2\lambda_2}{\alpha_2} + R_2\right) + \frac{R_2^{n+2}}{n+2}\right], \quad \eta \in (\delta;\ 0); \tag{20}$$

and for $t_2 = t_3$, $\Phi_2(\eta) = 0$

$$\tau = \frac{-\rho\gamma_2}{2\lambda_1 R_1^n\left(t_1 - t_3 + q_1\frac{1}{\alpha_1}\right)}\left[R_1^n\frac{\eta^2}{2} + \frac{(R_1+\eta)^{n+1}}{n+1} \times \right.$$

$$\left. \times\left(\frac{2\lambda_1}{\alpha_1} - R_1\right) + \frac{(R_1+\eta)^{n+2}}{n+2} - \frac{R_1^{n+1}}{n+1}\left(\frac{2\lambda_1}{\alpha_1} - R_1\right) - \frac{R_1^{n+2}}{n+2}\right], \quad \eta \in (0;\ \delta). \tag{21}$$

The time of total shell solidification τ_m is

$$\tau_m = \frac{\rho\gamma_1}{2\lambda_2 R_2^n\left(t_2 - t_3 + q_2\frac{1}{\alpha_2}\right)}\left[\frac{R_1^{n+1} - R_2^{n+1}}{n+1}\left(\frac{2\lambda_2}{\alpha_2} + R_2\right) - \frac{R_1^{n+2} - R_2^{n+2}}{n+2} - R_2^n\frac{\delta^2}{2}\right] \tag{22}$$

for $t_1 = t_3$ and

TABLE 1. Comparison between Obtained and Known [2] Approximate Solutions of the Solidification Problem

Body shape	Solution (24)	Solution [2]
Plate	$\tau_m = \frac{\delta^2}{2} \frac{\rho\gamma_1}{\lambda_2(t_3 - t_2)}$	$\tau_m = \frac{\delta^2}{2} \frac{\rho\gamma_1}{\lambda_2(t_3 - t_2)}$
Cylinder	$\tau_m = \frac{\delta^2}{3} \frac{\rho\gamma_1}{\lambda_2(t_3 - t_2)}$	$\tau_m = \frac{\delta^2}{4} \frac{\rho\gamma_1}{\gamma_2(t_3 - t_2)}$
Sphere	$\tau_m = \frac{7\delta^2}{24} \frac{\rho\gamma_1}{\lambda_2(t_3 - t_2)}$	$\tau_m = \frac{\delta^2}{6} \frac{\rho\gamma_1}{\lambda_2(t_3 - t_2)}$

$$\tau_m = \frac{-\rho\gamma_2}{2\lambda_1 R_1^n \left(t_1 - t_3 + q_1 \frac{1}{\alpha_1}\right)} \left[\frac{R_2^{n+1} - R_1^{n+1}}{n+1} \left(\frac{2\lambda_1}{\alpha_1} - R_1\right) + \frac{R_2^{n+2} - R_1^{n+2}}{n+2} + R_1^n \frac{\delta^2}{2}\right] \quad (23)$$

for $t_2 = t_3$.

The solutions (20) and (22) are easily carried over to the problem of the solidification of complex bodies ($R_1 = 0$, $R_2 = \delta$).

The solidification time of a complex one-dimensional body of complex shape is computed from the formula

$$\tau_m = \frac{-\rho\gamma_1}{\lambda_2 \left(t_2 - t_3 + q_2 \frac{1}{\alpha_2}\right)} \left[\delta^2 \frac{2 + (n+1)(n+2)}{4(n+1)(n+2)} + \delta \frac{1}{n+1} \frac{\lambda_2}{\alpha_2}\right]. \quad (24)$$

Recommendations on computing the quantities δ and n for a continuous homogeneous solid are given in [5]. For bodies of canonical shape the values of δ and n are determined as: δ is the half-thickness and n = 0 for a plate; δ is the radius and n = 1 for a cylinder, and δ is the radius and n = 2 for a sphere.

The possibility of analyzing the heat-exchange dynamics of one-dimensional continuous solids with the phase transition taken into account is confirmed by comparing the proposed solution (24) with the known solutions [2] for bodies of canonical shape obtained on the basis of the first method of L. S. Leibenzon for boundary conditions of the first kind (Table 1).

NOTATION

r is the running coordinate, $\bar{r}$, dimensionless coordinate; η, phase transition zone boundary; δ, governing dimension; n, form factor; τ, running time; τ_m, solidification time; λ, coefficient of heat conduction; and $S(\eta)$, phase interface surface area.

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